Mohua Banerjee
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Preface

The Indian Conference on Logic and Its Applications (ICLA) is a biennial conference organized under the aegis of the Association for Logic in India. The tenth edition of the conference was held during March 3–5, 2023, at the Indian Institute of Technology (IIT) Indore. This volume contains papers presented at the 10th ICLA.

A variety of themes are covered by the papers published in the volume. These are related to modal and temporal logics, intuitionistic connexive and imperative logics, systems for reasoning with vagueness and rough concepts, topological quasi-Boolean logic and quasi-Boolean based rough set models, and first-order definability of path functions of graphs. Three single blind reviews for each submission were ensured. Aside from reviews by the Program Committee (PC) members, there were reviews by external experts. In some cases, in order to reach a final decision on acceptance, there were further reviews by PC members or external experts. The Easy Chair system was used for submission and reviews; it proved to be quite convenient. We would like to express our deep appreciation to all the PC members for their efforts and support. We also thank all the external reviewers for their invaluable help.

ICLA 2023 included 8 invited talks, and 6 of these appear in the volume as full papers. We are immensely grateful to Mihir K. Chakraborty, Supratik Chakraborty, Marie Fortin, Giuseppe Greco, Kamal Lodaya, Sandra Müller, R. Ramanujam and Yde Venema for kindly accepting our invitations.

Special thanks are due to IIT Indore, the organizing committee steered by Md. Aquil Khan and all the volunteers, for making this edition of ICLA possible.

We are grateful to Springer, for agreeing to publish this volume in the LNCS series.

February 2023

Mohua Banerjee
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Quasi-Boolean Based Models in Rough Set Theory: A Case of Covering

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Abstract. Rough set theory has already been algebraically investigated for decades and quasi-Boolean algebra has formed a basis for several structures related to rough sets. An initiative has been taken in the paper [17] to obtain rough set models for some of these structures. These models have been constructed by defining a g-approximation space $\langle U, R^g \rangle$ out of a generalised approximation space $\langle U, R \rangle$ and an involution g. In this paper, as a continuation of [17], we have thrown light on covering cases and constructed a set model for the algebra IqBa2 [17].

Keywords: Rough set theory · Pre-rough algebra · Quasi-Boolean algebra · Modal logic

1 Introduction

Rough set theory has already been algebraically investigated for decades and quasi-Boolean algebra (qBa) has formed a base for a number of abstract algebras emerging out of rough sets [11]. Pre-rough algebra, amongst them, is one and it was defined by Banerjee and Chakraborty in [3]. The base of pre-rough algebra is a quasi-Boolean algebra which is a more general structure than a Boolean algebra as the law of excluded middle ($x \vee \sim x = 1$) and law of contradiction ($x \wedge \sim x = 0$) generally do not hold in a qBa. Topological quasi-Boolean algebra (tqBa) and topological quasi-Boolean algebra with modal axiom S_5 (tqBa5) come naturally as predecessors of pre-rough algebra.

Later, from different motivations, many abstract algebras stronger than qba but weaker than pre-rough algebra were developed. As for example, systemI algebra, systemII algebra [14] etc. have been introduced in order to access the rough implication $\rightarrow$ which was defined as $x \rightarrow y \equiv (\neg Ix \vee Iy) \wedge (\neg Cx \vee Cy)$ in pre-rough and rough algebras [2,3], where $C \equiv \neg I \neg$. On the other hand, three intermediate algebras IA1 (intermediate algebra of type 1), IA2 (intermediate algebra of type 2) and IA3 (intermediate algebra of type 3) [15,19] are defined based on three intermediate properties viz. $\neg Ix \vee Ix = 1$, for all x (IP1), $I(x \vee y) = Ix \vee Iy$, for all x, y (IP2) and $Ix \leq Iy$ and $Cx \leq Cy$ imply $x \leq y$, for all x, y (IP3) which play a crucial role to define rough implication.

Besides this, 3-valued Łukasiewicz (Monteiro) algebra [4], 3-valued Łukasiewicz (Moisil) algebra [5], Tetravalent Modal Algebra (TMA) [7] are some

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of the well-established algebraic structures based on quasi-Boolean algebra. It has been established in [1,14] that 3-valued Łukasiewicz (Monteiro) algebra and 3-valued Łukasiewicz (Moisil) algebra are equivalent to pre-rough algebra. Whereas in [14], it was observed that TMA is stronger than tqBa5 but weaker than a pre-rough algebra. In the same paper [14], it has been mentioned that the algebra MDS5 [6] is equivalent to IA2 if lattice distributivity is added to MDS5. A relationship diagram amongst the aforesaid algebras is shown in Fig. 1. For details of these algebras and their logics we refer to [2,3,14,18].

In the paper [3], a rough set model has been constructed for the abstract pre-rough algebra. It was developed in the context of rough set theory specially based on the notions of rough equality and rough inclusion. It has been described in [3] as follows. Let $\langle U, R\rangle$ be an approximation space. Two subsets P and Q of U are said to be roughly equal if $\underline{P}_R = \underline{Q}_R$ and $\overline{P}^R = \overline{Q}^R$ where $\underline{P}_R$ and $\overline{P}^R$ are Pawlakian lower and upper approximations of P respectively. An equivalence relation $\preccurlyeq$ is defined in 2^U, the power set of U, as $P \preccurlyeq Q$ if and only if P and Q are roughly equal. Each equivalence class $[P]_{\preccurlyeq}$ of $2^U/_{\preccurlyeq}$ is called a rough set. Using these rough sets and suitable operations $\sqcap, \sqcup, \neg$ and I, $\langle 2^U/_{\preccurlyeq}, \sqcap, \sqcup, \neg, I, [\emptyset]_{\preccurlyeq}, [U]_{\preccurlyeq}\rangle$ is a model of an abstract pre-rough algebra. The operations $\sqcap, \sqcup, \neg$ and I are defined as

$$
\begin{aligned}
&[P]_{\preccurlyeq} \sqcap [Q]_{\preccurlyeq} = [P \sqcap Q]_{\preccurlyeq},\\
&[P]_{\preccurlyeq} \sqcup [Q]_{\preccurlyeq} = [P \sqcup Q]_{\preccurlyeq},\\
&\neg[P]_{\preccurlyeq} = [\neg P]_{\preccurlyeq},\\
&I[P]_{\preccurlyeq} = [IP]_{\preccurlyeq},
\end{aligned}
$$

where

$$
\begin{aligned}
&P \sqcap Q = (P \cap Q) \cup (P \cap \overline{Q}^R \cap (\overline{P \cap Q}^R)^c),\\
&P \sqcup Q = (P \cup Q) \cap (P \cup \underline{Q}_R \cup (\underline{P \cup Q}_R)^c),\\
&\neg P = P^c,\\
&IP = \underline{P}_R,
\end{aligned}
$$

$\cap, \cup$ and c being the set theoretic intersection, union and complementation. The lattice order $\sqsubseteq$ in the above pre-rough algebra is $[P]_{\preccurlyeq} \sqsubseteq [Q]_{\preccurlyeq}$ if and only if P is roughly included in Q, i.e., $\underline{P}_R \subseteq \underline{Q}_R$ and $\overline{P}^R \subseteq \overline{Q}^R$.

But, there are no proper set theoretic rough set models of the abstract algebras shown in Fig. 1 which are really weaker than pre-rough algebras. The phrase 'proper set theoretic rough set model' means that it should be a set model and should not reduce to a pre-rough algebra. A step has been taken in this regard in the paper [18]. In this paper, set models of System0, stqBa, stqBa-D, stqBa-T, stqBa-B, tqBa, tqBa5 and IA1 have been developed using the relation-based rough set theory.

Another direction of work was initiated in the papers [15,17]. In these papers, the authors have considered those algebras where an implication ($\rightarrow$) satisfying the property ($P_\rightarrow$): $x \leq y$ if and only if $x \rightarrow y = 1$, for all x, y, can not be defined or not available till now. It is to be noted that an implication $\rightarrow$ satisfying the property ($P_\rightarrow$) is required in an algebra to develop the Hilbert-type logic system corresponding to the algebra. For construction of the said logic system, following Rasiowa, algebras are defined by imposing an implication $\rightarrow$ obeying the property ($P_\rightarrow$). These algebras are shown in Fig. 2 and for further information about the algebras and their logics one may see the papers [15,17]. Rough set models of some of the algebras IqBaO, IqBaT, IqBa4, IqBa5, IqBa1, IqBa1,T, IqBa1,4 and IqBa1,5 have been presented in [17].

This current paper deals with a parallel type of research that has been initiated in our earlier papers [17,18]. In fact, in this paper, covering cases are considered and one set model has been developed using "deleted neighborhood", in other words, anti-reflexive neighborhood that has importance in a number of areas of computer applications, e.g., the field of computer security [9].

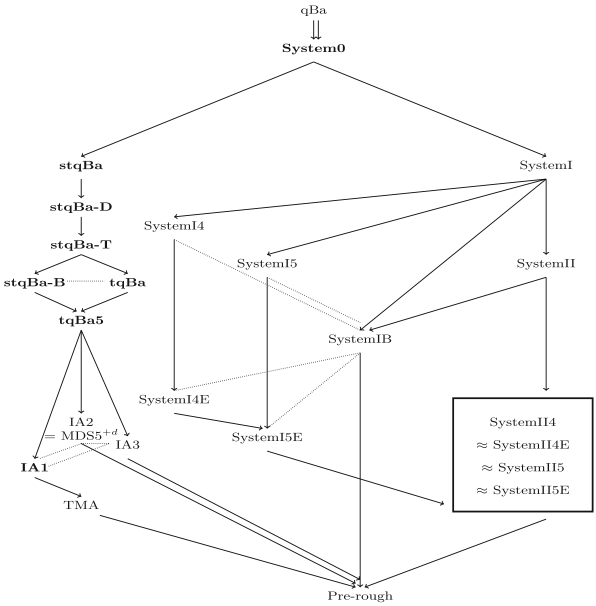

Fig. 1. Structures based on qBa: $P \rightrightarrows Q$ stands for the algebra Q has one more operator and some axioms for the new operator than the algebra P. $P \longrightarrow Q$ stands for both the algebras P and Q have the same operations and the algebra Q is always the algebra P. $P \cdots Q$ stands for the algebras P and Q are independent.

2 Rough Set Models - Relational Approach

In the papers [17,18], rough set models have been presented for the algebras System0, stqBa, stqBa-D, stqBa-T, stqBa-B, tqBa, tqBa5, IA1, IqBaO, IqBaT,

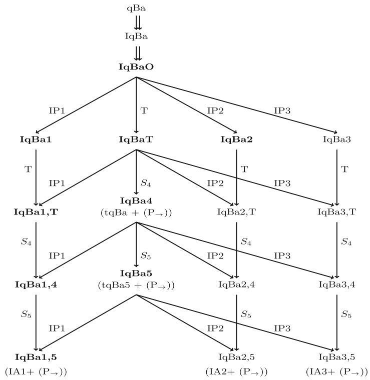

Fig. 2. Algebras with imposed implications: $P \rightrightarrows Q$ stands for the algebra Q has one more operator and some axioms for the new operator than the algebra P. $P \longrightarrow Q$ stands for both the algebras P and Q have the same operations and the algebra Q is always the algebra P.

IqBa4, IqBa5, IqBa1, IqBa1,T, IqBa1,4 and IqBa1,5. All these algebras are based on qBa and therefore to construct their rough set models we have focused our attention on a representation theorem of qBa developed by Rasiowa [13]. As demonstrated by her, for any set U we can define an algebra $\langle 2^U, \cap, \cup, \neg, \emptyset, U\rangle$ which may be proved to be a quasi Boolean algebra, where $\neg$, called quasi-complementation, is not the standard set-theoretic complementation ^{c} but is defined by means of an involution g (i.e. a map on U satisfying $g(g(u)) = u$, for all $u \in U$) namely $\neg P = (g(P))^c$, $P \subseteq U$. The lower and the upper approximation operators $\underline{\ }_R, \overline{\ }^R : 2^U \to 2^U$ have been defined in a generalised approximation space $\langle U, R\rangle$ by

$$\underline{P}_R = \{u \in U : R_u \subseteq P\}$$

and

$$\overline{P}^R = \{u \in U : R_u \cap P \neq \emptyset\},$$

where $R_u = \{v \in U : uRv\}$. For any $P \in 2^U$, $\underline{P}_R$ and $\overline{P}^R$ are dual with respect to the set complementation; the question is, how to define the algebraic counterparts of these operators in the aforementioned quasi-Boolean algebra, so as to make them dual with respect to the quasi-complementation $\neg$. The issue has been resolved by defining a g-approximation space $\langle U, R^g\rangle$ out of a generalised approximation space $\langle U, R\rangle$ and an involution g on U.

Let $\langle U, R\rangle$ be a generalised approximation space and $g : U \to U$ be an involution. A binary relation R^g on U has been defined as follows:

$$\text{for any two elements } u \text{ and } v \in U, uR^g v \text{ if and only if } g(u)Rg(v). \quad (1)$$

That is, two elements $u, v \in U$ are related with respect to a new relation R^g if and only if their g-images are related in the relation R. We call $\langle U, R^g\rangle$ a g-generalised approximation space or simply, a g-approximation space.

As g is an involution on U, R can be obtained from R^g as follows:

$$\text{for any two elements } u \text{ and } v \in U, uRv \text{ if and only if } g(u)R^g g(v). \tag{2}$$

Similarly, it says that two elements $u, v \in U$ will be related in the relation R if and only if their g-images are so in the relation R^g.

In this g-approximation space $\langle U, R^g\rangle$, we have defined g-lower approximation and g- upper approximation $\underline{\ }_g, \overline{\ }^g : 2^U \to 2^U$ as follows:
for any $P \in 2^U$,

$$\underline{P}_g = \{u \in U : R_u^g \subseteq P\}$$

and

$$\overline{P}^g = \{u \in U : R_{g(u)}^g \cap g(P) \neq \emptyset\}$$

where $R_u^g = \{v \in U : uR^g v\}$. Using these lower-upper approximations and imposing conditions like reflexivity, symmetric, transitivity etc. on R^g proper rough set models of System0, stqBa, stqBa-D, stqBa-T, stqBa-B, tqBa, tqBa5, IA1 have been constructed in [18].

To construct rough set models for the algebras IqBaO, IqBaT, IqBa4, IqBa5 shown in Fig. 2, a suitable operation that corresponds to $\to$ (available in the above algebras) is needed. Boolean implication $P \to Q(\equiv P^c \cup Q)$, in one way, serves the purpose smoothly. On the other hand, g image of Boolean implication $g(P \to Q)(\equiv P \to_1 Q)$ also fulfils the property ($\mathrm{P}_\to$). With their help, rough set models of IqBaO, IqBaT, IqBa4, IqBa5 have been presented in [17].

A pair of new approximation operators $\underline{\ }_{g,1}, \overline{\ }^{g,1} : 2^U \to 2^U$ has been defined [17] in order to obtain set models for the algebras IqBa1, IqBa1,T, IqBa1,4 and IqBa1,5 as follows:

$$\underline{P}_{g,1} = \{u \in U : R_u^g \subseteq P\} \cap \{u \in U : R_{g(u)}^g \subseteq P\}$$

and

$$\overline{P}^{g,1} = \{u \in U : R_{g(u)}^g \cap g(P) \neq \emptyset\} \cup \{u \in U : R_u^g \cap g(P) \neq \emptyset\}.$$

For details, one may see the paper [17].

3 Rough Set Model - Covering Based Approach

In this section we shall discuss the covering based rough sets and incorporate the involution g to construct lower-upper approximations so that they will be dual approximations with respect to the quasi-complementation. As we have constructed two types of lower-upper approximations based on a binary relation, some natural questions may arise on covering cases in the following form:

- How can a parallel study be introduced on covering based rough set theory and what would be outcomes in that case?
- Is it possible to develop rough set models of some of the remaining algebras through this study?

In response to the first question, we have defined a g-covering approximation space out of a covering approximation space and an involution g. Thereafter, the basic notions like $Friends$ of u, $Neighborhood$ of u etc. are introduced in a g-covering approximation space in the same way as they have been defined in a covering approximation space. Relationships between these two spaces and the above-mentioned notions are studied.

For the last question, a new type of collection at each point of a g-covering approximation space has been developed. We call it a "deleted neighborhood". For the importance of this neighborhood, we have taken the following words as it is from the paper [9]: "a neighborhood N(p) of p may be punctured or empty; by that we mean the neighborhood does not contain its center p or is an empty set. Such a neighborhood is called an anti-reflexive neighborhood, including the case of empty neighborhood. It is useful in many applications, e.g., in computer security. We may consider a set of "my" enemies as a neighborhood. Surely, "myself" is not included in that set".

With the help of this deleted neighborhood or anti-reflexive neighborhood, lower-upper approximations have been defined. A rough set model of IqBa2 has been presented using these lower-upper approximations.

3.1 Basics in a Covering Approximation Space

Definition 1 [16] *(Covering of a set): Let U be a non empty set and $\mathcal{C} = \{U_i(\neq \emptyset) \subseteq U : i \in I\}$, where I is an index set, is said to be a covering of U if*

$$\underset{i\in I}{\cup} U_i = U.$$

Definition 2 [16] *(Covering approximation space): Let U be a non empty set and $\mathcal{C}$ be a covering of U. Then, the ordered pair $\langle U, \mathcal{C}\rangle$ is called a covering approximation space.*

Definition 3. *Let $\langle U, \mathcal{C}\rangle$ be a covering approximation space. For each $u \in U$,*

1. *(Friends of u): [16] Friends of u is defined by*

$$F^{\mathcal{C}}(u) = \underset{u\in U_i}{\cup} U_i.$$

   It is also called the indiscernible neighborhood of u [10].

2. *(Neighborhood of u): [16] Neighborhood of u is defined by*

$$N^{\mathcal{C}}(u) = \underset{u\in U_i}{\cap} U_i.$$

3. *(Friends' enemy of u): [10,16] Friends' enemy of u is defined by*

$$FE^{\mathcal{C}}(u) = U - F^{\mathcal{C}}(u).$$

4. *(Kernel of u):* [16] *Kernel of u is defined by*

$$K^{\mathcal{C}}(u) = \{y \in U : \forall U_i(u \in U_i \Leftrightarrow y \in U_i)\}.$$

Let $\mathcal{P}^{\mathcal{C}} = \{K^{\mathcal{C}}(u) : u \in U\}$. Then, $\mathcal{P}^{\mathcal{C}}$ is a partition of U and called a partition generated by the covering $\mathcal{C}$.

5. *(Minimal description and Maximal description of u):* [12,20] *Minimal description and Maximal description of u are defined respectively as*

$$md^{\mathcal{C}}(u) = \{U_i \in \mathcal{C} : u \in U_i \text{ and } \forall U \in \mathcal{C}(u \in U \subseteq U_i \text{ implies } U = U_i)\}$$

and

$$Md^{\mathcal{C}}(u) = \{U_i \in \mathcal{C} : u \in U_i \text{ and } \forall U \in \mathcal{C}(U \supseteq U_i \text{ implies } U = U_i)\}.$$

We are now going to define a g-covering approximation space in the following way.

3.2 g-covering Approximation Space

Proposition 1. *Let $\langle U, \mathcal{C}\rangle$ be a covering approximation space and $g\colon U \to U$ be an involution, i.e., $g(g(u)) = u$, for all $u \in U$. Then $g(\mathcal{C}) = \{g(U_i) : U_i \in \mathcal{C}\}$ is a covering of U.*

Proof. Let $u \in U$. Then, $g(u) \in U_i$, for some $i \in I$ (As, $\mathcal{C} = \{U_i(\neq \emptyset) \subseteq U : i \in I\}$ is a covering of U). Then, by the definition of g, $u \in g(U_i)$ and hence $g(\mathcal{C}) = \{g(U_i) : U_i \in \mathcal{C}\}$ is a covering of U.

From the above proposition, we now define a g-covering approximation space below.

Definition 4. *Let $\langle U, \mathcal{C}\rangle$ be a covering approximation space and g be an involution on U. Then, $\langle U, g(\mathcal{C})\rangle$ will be called a g-covering approximation space.*

In general, $\mathcal{C} \neq g(\mathcal{C})$. The following example supports the statement.

Example 1. Let $U = \{a, b, c, d, e\}$, $\mathcal{C} = \{U_1 = \{a, b\}, U_2 = \{d, e\}, U_3 = \{c, e\}\}$ be a covering of U and $g : U \to U$ be an involution defined by $g(a) = c, g(b) = d, g(c) = a, g(d) = b, g(e) = e$. Now, g($\mathcal{C}) = \{g(U_1) = \{c, d\}, g(U_2) = \{b, e\}, g(U_3) = \{a, e\}\}$ and hence $\mathcal{C} \neq g(\mathcal{C})$.

The following is a necessary and sufficient condition that reveals when $\mathcal{C}$ and $g(\mathcal{C})$ coincide.

Proposition 2. *Let $\langle U, g(\mathcal{C})\rangle$ be a g-covering approximation space. Then $\mathcal{C} = g(\mathcal{C})$ if and only if for each $i \in I$, $g(U_i) = U_j$, for some $j \in I$.*

Proof. Let $\mathcal{C} = g(\mathcal{C})$ and $U_i \in \mathcal{C}$, for any $i \in I$. Then $U_i \in g(\mathcal{C})$ [as $\mathcal{C} = g(\mathcal{C})$]. This gives, $U_i = g(U_j)$, for some $j \in I$. Conversely, let for each $i \in I$ there exist $j \in I$ such that $g(U_i) = U_j$. We have to show that $\mathcal{C} = g(\mathcal{C})$. Let $U_i \in \mathcal{C}$. Then by the hypothesis $g(U_i) = U_j$, for some $j \in I$. Then by the definition of g, $U_i = g(U_j)$. As $U_j \in \mathcal{C}$, $g(U_j) \in g(\mathcal{C})$, i.e., $U_i \in g(\mathcal{C})$. Thus, $\mathcal{C} \subseteq g(\mathcal{C})$. Let $Y \in g(\mathcal{C})$. Then $Y = g(U_j)$, for some $U_j \in \mathcal{C}$. Then by the hypothesis $g(U_j) = U_k$, for some $k \in I$. Thus, $Y = U_k \in \mathcal{C}$ and hence $g(\mathcal{C}) \subseteq \mathcal{C}$.

Note 1. If $g(U_i) = U_i$, for all $i \in I$ then $\mathcal{C} = g(\mathcal{C})$. But the converse, i.e., $\mathcal{C} = g(\mathcal{C})$ implies $g(U_i) = U_i$, for all $i \in I$, is not true as shown by an example given below.

Example 2. Let U and g be the same as mentioned in Example 1. Let $\mathcal{C} = \{U_1 = \{a, e\}, U_2 = \{c, e\}, U_3 = \{b\}, U_4 = \{d\}\}$ be a covering of U. Then, g($\mathcal{C}$) $= \{g(U_1) = \{c, e\}, g(U_2) = \{a, e\}, g(U_3) = \{d\}, g(U_4) = \{b\}\}$ and hence $\mathcal{C} = g(\mathcal{C})$ but for none of $i = 1, 2, 3, 4$, $g(U_i) = U_i$.

Now, we define the notions of *Friends* of u, *Neighborhood* of u etc. in a g-covering approximation space $\langle U, g(\mathcal{C})\rangle$.

Definition 5. *Let $\langle U, g(\mathcal{C})\rangle$ be a g-covering approximation space. Then for each $u \in U$,*

1. *Friends of u is defined by*
$$F^{g(\mathcal{C})}(u) = \bigcup_{u \in g(U_i)} g(U_i),$$
2. *Neighborhood of u is defined by*
$$N^{g(\mathcal{C})}(u) = \bigcap_{u \in g(U_i)} g(U_i),$$
3. *Friends' enemy of u is defined by*
$$FE^{g(\mathcal{C})}(u) = U - F^{g(\mathcal{C})}(u),$$
4. *Kernel of u is defined by*
$$K^{g(\mathcal{C})}(u) = \{y \in U : \forall g(U_i)(u \in g(U_i) \Leftrightarrow y \in g(U_i))\}.$$
Let $\mathcal{P}^{g(\mathcal{C})} = \{K^{g(\mathcal{C})}(u) : u \in U\}$. Then, $\mathcal{P}^{g(\mathcal{C})}$ is a partition of U and hence it will be called partition generated by the covering $g(\mathcal{C})$.
5. *Minimal description of u is defined by* $md^{g(\mathcal{C})}(u) = \{g(U_i) \in g(\mathcal{C}) : u \in g(U_i)$ *and* $\forall X \in g(\mathcal{C})(u \in X \subseteq g(U_i)$ *implies* $X = g(U_i))\}$.
6. *Maximal description of u is defined by* $Md^{g(\mathcal{C})}(u) = \{g(U_i) \in g(\mathcal{C}) : u \in g(U_i)$ *and* $\forall X \in g(\mathcal{C})(X \supseteq g(U_i)$ *implies* $X = g(U_i))\}$.

The following example is considered to show that *Friends* of u, *Neighborhood* of u etc. in a covering approximation space are generally not the same with *Friends* of u, *Neighborhood* of u etc. in the g-covering approximation space.

Example 3. Let $U, \mathcal{C}$ and g be the same as defined in Example 1. Considering $u = c$ we get

1. $F^{g(\mathcal{C})}(c) = g(U_1) = \{c, d\} \neq F^{\mathcal{C}}(c) = \{c, e\}$,
2. $N^{g(\mathcal{C})}(c) = g(U_1) = \{c, d\} \neq N^{\mathcal{C}}(c) = \{c, e\}$,
3. $FE^{g(\mathcal{C})}(c) = U - F^{g(\mathcal{C})}(c) = \{a, b, e\} \neq FE^{\mathcal{C}}(c) = \{a, b, d\}$,

4. $K^{g(\mathcal{C})}(c) = \{y \in U : \forall g(U_i)(c \in g(U_i) \Leftrightarrow y \in g(U_i))\} = \{c, d\} \neq K^{\mathcal{C}}(c) = \{c\}$,
5. $md^{g(\mathcal{C})}(c) = \{g(U_i) \in g(\mathcal{C}) : c \in g(U_i)$ and $\forall X \in g(\mathcal{C})(c \in X \subseteq g(U_i)$ implies $X = g(U_i))\} = \{g(U_1) = \{c, d\}\} \neq md^{\mathcal{C}}(c) = \{U_3 = \{c, e\}\}$,
6. $Md^{g(\mathcal{C})}(c) = \{g(U_i) \in g(\mathcal{C}) : c \in g(U_i)$ and $\forall X \in g(\mathcal{C})(U \supseteq g(U_i)$ implies $X = g(U_i))\} = \{g(U_1) = \{c, d\}\} \neq Md^{\mathcal{C}}(c) = \{U_3 = \{c, e\}\}$.
7. $\mathcal{P}^{g(\mathcal{C})} = \{K^{g(\mathcal{C})}(u) : u \in U\} = \{K^{g(\mathcal{C})}(a) = \{a\}, K^{g(\mathcal{C})}(b) = \{b\}, K^{g(\mathcal{C})}(c) = \{c, d\} = K^{g(\mathcal{C})}(d), K^{g(\mathcal{C})}(e) = \{e\}\} \neq \mathcal{P}^{\mathcal{C}} = \{K^{\mathcal{C}}(a) = \{a, b\} = K^{\mathcal{C}}(b), K^{\mathcal{C}}(c) = \{c\}, K^{\mathcal{C}}(d) = \{d\}, K^{g(\mathcal{C})}(e) = \{e\}\}$

Proposition 3. *Let $\langle U, g(\mathcal{C})\rangle$ be a g-covering approximation space. Then,*

1. *$F^{\mathcal{C}}(u) = g(F^{g(\mathcal{C})}(g(u)))$ and $F^{g(\mathcal{C})}(u) = g(F^{\mathcal{C}}(g(u)))$, for all $u \in U$,*
2. *$N^{\mathcal{C}}(u) = g(N^{g(\mathcal{C})}(g(u)))$ and $N^{g(\mathcal{C})}(u) = g(N^{\mathcal{C}}(g(u)))$, for all $u \in U$,*
3. *$FE^{\mathcal{C}}(u) = g(FE^{g(\mathcal{C})}(g(u)))$ and $FE^{g(\mathcal{C})}(u) = g(FE^{\mathcal{C}}(g(u)))$, for all $u \in U$,*
4. *$K^{\mathcal{C}}(u) = g(K^{g(\mathcal{C})}(g(u)))$ and $K^{g(\mathcal{C})}(u) = g(K^{\mathcal{C}}(g(u)))$, for all $u \in U$,*
5. *$md^{\mathcal{C}}(u) = g(md^{g(\mathcal{C})}(g(u)))$ and $md^{g(\mathcal{C})}(u) = g(md^{\mathcal{C}}(g(u)))$, for all $u \in U$,*
6. *$Md^{\mathcal{C}}(u) = g(Md^{g(\mathcal{C})}(g(u)))$ and $Md^{g(\mathcal{C})}(u) = g(Md^{\mathcal{C}}(g(u)))$, for all $u \in U$,*
7. *$\mathcal{P}^{\mathcal{C}} = g(\mathcal{P}^{g(\mathcal{C})})$ and $\mathcal{P}^{g(\mathcal{C})} = g(\mathcal{P}^{\mathcal{C}})$, where $g(\mathcal{P}^{g(\mathcal{C})}) = \{g(Y) : Y \in \mathcal{P}^{g(\mathcal{C})}\}$ and similarly for $g(\mathcal{P}^{\mathcal{C}})$.*

Proof. 1. Let $y \in F^{\mathcal{C}}(u)$. Then, $y, u \in U_j$, for some $j \in I$ and hence $g(y), g(u) \in g(U_j)$. This gives, $g(y) \in F^{g(\mathcal{C})}(g(u))$ and hence $g(g(y)) \in g(F^{g(\mathcal{C})}(g(u)))$, i.e., $y \in g(F^{g(\mathcal{C})}(g(u)))$. Thus $F^{\mathcal{C}}(u) \subseteq g(F^{g(\mathcal{C})}(g(u)))$. Let $y \in g(F^{g(\mathcal{C})}(g(u)))$. Then $y = g(z)$, where $z \in F^{g(\mathcal{C})}(g(u))$. This implies, $z, g(u) \in g(U_k)$, for some $k \in I$ and therefore $g(z),\ g(g(u)) \in g(g(U_k))$, i.e., $y = g(z),\ u \in U_k$. This gives, $y \in F^{\mathcal{C}}(u)$ and therefore $g(F^{g(\mathcal{C})}(g(u))) \subseteq F^{\mathcal{C}}(u)$. Thus, $F^{\mathcal{C}}(u) = g(F^{g(\mathcal{C})}(g(u)))$. Proofs of 2, 3, 4, 5, 6 and 7 can be done similarly.

It is time now to define deleted neighborhood or anti-reflexive neighborhood of an element u in U in order to develop a rough set model for the algebra IqBa2.

Definition 6. *Let $\langle U, g(\mathcal{C})\rangle$ be a g-covering approximation space. For each $u \in U$, deleted Neighbourhood of u in the covering approximation space $\langle U, \mathcal{C}\rangle$ and in the g-covering approximation space $\langle U, g(\mathcal{C})\rangle$, denoted by $N_d^{\mathcal{C}}(u)$ and $N_d^{g(\mathcal{C})}(u)$ respectively, are defined by $N_d^{\mathcal{C}}(u) = N^{\mathcal{C}}(u) - \{u\}$ and $N_d^{g(\mathcal{C})}(u) = N^{g(\mathcal{C})}(u) - \{u\}$.*

Note 2. For each $u \in U$, u does not belong to $N_d^{\mathcal{C}}(u)$ and $N_d^{g(\mathcal{C})}(u)$. Moreover, $N_d^{\mathcal{C}}(u)$ or $N_d^{g(\mathcal{C})}(u)$ may be empty for some $u \in U$.

Proposition 4. *Let $\langle U, g(\mathcal{C})\rangle$ be a g-covering approximation space. Then for each $u \in U$, $N_d^{\mathcal{C}}(u) = g(N_d^{g(\mathcal{C})}(g(u)))$ and $N_d^{g(\mathcal{C})}(u) = g(N_d^{\mathcal{C}}(g(u)))$.*

Proof.

$$\begin{aligned}
N_d^{g(\mathcal{C})}(g(u)) &= N^{g(\mathcal{C})}(g(u)) - \{g(u)\} \text{ [from Definition 6]}\\
\text{Then, } g(N_d^{g(\mathcal{C})}(g(u))) &= g(N^{g(\mathcal{C})}(g(u)) - \{g(u)\})\\
&= g(N^{g(\mathcal{C})}(g(u))) - g(\{g(u)\}) \text{ [as } g(A - B) = g(A) - g(B)\text{]}\\
&= N^{\mathcal{C}}(u) - \{u\} \text{ [by 2 of Proposition 3]}\\
&= N_d^{\mathcal{C}}(u) \text{ [from Definition 6]}
\end{aligned}$$

Similarly, the other part can be proved.

3.3 Rough Set Model for IqBa2

For a set-theoretic rough set model of the algebra IqBa2, we have to develop a pair of lower-upper approximations which must be dual with respect to the quasi-complementation and satisfies the property IP2: $I(a \vee b) = Ia \vee Ib$. Due to this reason, we define a new pair of lower-upper approximations as follows.

Definition 7. *Let $\langle U, g(\mathcal{C})\rangle$ be a g-covering approximation space. Then for any subset A of U, $\underline{A}_{g(\mathcal{C}),2}$, the $g,2$ lower approximation of A and $\overline{A}^{g(\mathcal{C}),2}$, the $g,2$ upper approximation of A are defined by*

$$\underline{A}_{g(\mathcal{C}),2} = \{u \in U : N_d^{g(\mathcal{C})}(u) \subseteq A\} \tag{3}$$

and

$$\overline{A}^{g(\mathcal{C}),2} = \{u \in U : N_d^{\mathcal{C}}(u) \cap A \neq \emptyset\}. \tag{4}$$

Proposition 5. *In a g-covering approximation space $\langle U, g(\mathcal{C})\rangle$, $\underline{A}_{g(\mathcal{C}),2}$ and $\overline{A}^{g(\mathcal{C}),2}$ are dual approximations with respect to the quasi-complementation $\neg$ defined through g.*

Proof.

$$\begin{aligned}
\neg\left(\underline{\neg A}_{g(\mathcal{C}),2}\right) &= \neg\left(\underline{g(A)^c}_{g(\mathcal{C}),2}\right) \text{ [as } \neg A = (g(A))^c\text{]} \\
&= \neg\{u \in U : N_d^{g(\mathcal{C})}(u) \subseteq g(A)^c\} \text{ [by Definition 7]} \\
&= U - \{g(u) : N_d^{g(\mathcal{C})}(u) \subseteq g(A)^c\} \text{ [as } \neg A = U - g(A)\text{]} \\
&= U - \{u \in U : N_d^{g(\mathcal{C})}(g(u)) \subseteq g(A)^c\} \text{ [taking } g(u) \text{ as } u\text{]} \\
&= \{u \in U : N_d^{g(\mathcal{C})}(g(u)) \cap g(A) \neq \emptyset\} \\
&= \{u \in U : g(N_d^{\mathcal{C}}(u)) \cap g(A) \neq \emptyset\} \text{ [by Proposition 4]} \\
&= \{u \in U : g(N_d^{\mathcal{C}}(u) \cap A) \neq \emptyset\} \text{ [as } g(A \cap B) = g(A) \cap g(B)\text{]} \\
&= \{u \in U : N_d^{\mathcal{C}}(u) \cap A \neq \emptyset\} \text{ [as } g \text{ is an involution]} \\
&= \overline{A}^{g(\mathcal{C}),2}.
\end{aligned}$$

As $\neg\neg A = A$, hence $\underline{A}_{g(\mathcal{C}),2}$ and $\overline{A}^{g(\mathcal{C}),2}$ are dual approximations with respect to the quasi-complementation $\neg$ defined through g.

Proposition 6. *In a g-covering approximation space $\langle U, g(\mathcal{C})\rangle$, the following results hold.*

1. *$\underline{X}_{g(\mathcal{C}),2} = U$ and $\overline{\emptyset}^{g(\mathcal{C}),2} = \emptyset$.*
2. *If $A \subseteq B \subseteq U$ then $\underline{A}_{g(\mathcal{C}),2} \subseteq \underline{B}_{g(\mathcal{C}),2}$ and $\overline{A}^{g(\mathcal{C}),2} \subseteq \overline{B}^{g(\mathcal{C}),2}$.*
3. *$\underline{A \cap B}_{g(\mathcal{C}),2} = \underline{A}_{g(\mathcal{C}),2} \cap \underline{B}_{g(\mathcal{C}),2}$ and $\overline{A \cup B}^{g(\mathcal{C}),2} = \overline{A}^{g(\mathcal{C}),2} \cup \overline{B}^{g(\mathcal{C}),2}$, for all $A, B \subseteq U$.*

Proof. Proof of 1 is straightforward.
For proof of 2, let $x \in \underline{A}_{g(\mathcal{C}),2}$. Then by Definition 7, $N_d^{g(\mathcal{C})}(x) \subseteq A$ and hence $N_d^{g(\mathcal{C})}(x) \subseteq B$ (as $A \subseteq B$). This gives, $x \in \underline{B}_{g(\mathcal{C}),2}$ and hence $\underline{A}_{g(\mathcal{C}),2} \subseteq \underline{B}_{g(\mathcal{C}),2}$. Similarly, $\overline{A}^{g(\mathcal{C}),2} \subseteq \overline{B}^{g(\mathcal{C}),2}$ holds.
Proof of 3: To show $\underline{A \cap B}_{g(\mathcal{C}),2} = \underline{A}_{g(\mathcal{C}),2} \cap \underline{B}_{g(\mathcal{C}),2}$, we have to prove $\underline{A}_{g(\mathcal{C}),2} \cap \underline{B}_{g(\mathcal{C}),2} \subseteq \underline{A \cap B}_{g(\mathcal{C}),2}$. Let $x \in \underline{A}_{g(\mathcal{C}),2} \cap \underline{B}_{g(\mathcal{C}),2}$. Then, $N_d^{g(\mathcal{C})}(x) \subseteq A$ and B. Therefore, $N_d^{g(\mathcal{C})}(x) \subseteq A \cap B$ and hence $x \in \underline{A \cap B}_{g(\mathcal{C}),2}$. Thus, $\underline{A}_{g(\mathcal{C}),2} \cap \underline{B}_{g(\mathcal{C}),2} \subseteq \underline{A \cap B}_{g(\mathcal{C}),2}$ and hence the result is proved. Similarly, the other part of 3 can be proved.

Theorem 1. *In a g-covering approximation space $\langle U, g(\mathcal{C})\rangle$, $\underline{A \cup B}_{g(\mathcal{C}),2} = \underline{A}_{g(\mathcal{C}),2} \cup \underline{B}_{g(\mathcal{C}),2}$ holds for all $A, B \subseteq U$ if and only if for each $u \in U$, $N_d^{g(\mathcal{C})}(u)$ contains at most one element of U.*

Proof. Let $\underline{A \cup B}_{g(\mathcal{C}),2} = \underline{A}_{g(\mathcal{C}),2} \cup \underline{B}_{g(\mathcal{C}),2}$, for all $A, B \subseteq U$. It is to be proved that $N_d^{g(\mathcal{C})}(u)$ contains at most one element of U. If possible, let $N_d^{g(\mathcal{C})}(u)$ contain more than one element of U. Then, there are at least two distinct elements $y, z \in N_d^{g(\mathcal{C})}(u)$ where $y \neq u$ and $z \neq u$ [as $u \notin N_d^{g(\mathcal{C})}(u)$]. Let $A = \{y\}$ and $B = N_d^{g(\mathcal{C})}(u) - \{y\}$. Then $z \in B \neq \emptyset$. Then by hypothesis, $\underline{A \cup B}_{g(\mathcal{C}),2} = \underline{A}_{g(\mathcal{C}),2} \cup \underline{B}_{g(\mathcal{C}),2}$ holds, where $A = \{y\}$ and $B = N_d^{g(\mathcal{C})}(u) - \{y\}$. This gives, $\underline{N_d^{g(\mathcal{C})}(u)}_{g(\mathcal{C}),2} = \underline{A}_{g(\mathcal{C}),2} \cup \underline{B}_{g(\mathcal{C}),2}$. As $N_d^{g(\mathcal{C})}(u)$ is a subset of itself so $u \in \underline{N_d^{g(\mathcal{C})}(u)}_{g(\mathcal{C}),2} = \underline{A}_{g(\mathcal{C}),2} \cup \underline{B}_{g(\mathcal{C}),2}$. This implies, either $u \in \underline{A}_{g(\mathcal{C}),2}$ or $u \in \underline{B}_{g(\mathcal{C}),2}$, i.e., either $N_d^{g(\mathcal{C})}(u) \subseteq \{y\}$ or $N_d^{g(\mathcal{C})}(u) \subseteq N_d^{g(\mathcal{C})}(u) - \{y\}$. But we have $z \in N_d^{g(\mathcal{C})}(u) \nsubseteq \{y\}$ and $y \in N_d^{g(\mathcal{C})}(u) \nsubseteq N_d^{g(\mathcal{C})}(u) - \{y\}$. Thus, $N_d^{g(\mathcal{C})}(u)$ contains at most one element of U, for all $u \in U$.
Conversely, let us assume that each $N_d^{g(\mathcal{C})}(u)$ contains at most one element of U. We have to prove that $\underline{A \cup B}_{g(\mathcal{C}),2} = \underline{A}_{g(\mathcal{C}),2} \cup \underline{B}_{g(\mathcal{C}),2}$ holds for all $A, B \subseteq U$. By 2 of Proposition 6, it is sufficient to show that $\underline{A \cup B}_{g(\mathcal{C}),2} \subseteq \underline{A}_{g(\mathcal{C}),2} \cup \underline{B}_{g(\mathcal{C}),2}$. Let $u \in \underline{A \cup B}_{g(\mathcal{C}),2}$. Then, $N_d^{g(\mathcal{C})}(u) \subseteq A \cup B$. As $N_d^{g(\mathcal{C})}(u)$ contains at most one element, so, it follows that either $N_d^{g(\mathcal{C})}(u) \subseteq A$ or $N_d^{g(\mathcal{C})}(u) \subseteq B$ and hence $u \in \underline{A}_{g(\mathcal{C}),2} \cup \underline{B}_{g(\mathcal{C}),2}$. Thus, $\underline{A \cup B}_{g(\mathcal{C}),2} = \underline{A}_{g(\mathcal{C}),2} \cup \underline{B}_{g(\mathcal{C}),2}$, for all $A, B \subseteq U$.

Remark 1. As $\underline{A}_{g(\mathcal{C}),2}$ and $\overline{A}^{g(\mathcal{C}),2}$ are dual approximations with respect to the quasi-complementation $\neg$ and $A \cap B = \neg(\neg A \cup \neg B)$ so, $\overline{A \cap B}^{g(\mathcal{C}),2} = \overline{A}^{g(\mathcal{C}),2} \cap \overline{B}^{g(\mathcal{C}),2}$ holds for all $A, B \subseteq U$ if and only if for each $u \in U$, $N_d^{g(\mathcal{C})}(u)$ contains at most one element of U.

The following example is considered to show that $\underline{A}_{g(\mathcal{C}),2}$ may not be a subset of A, for some $A \subseteq U$.

Example 4. Let U, g and $\mathcal{C}$ be the same as defined in Example 1. Then, $N_d^{g(\mathcal{C})}(a) = \{e\}, N_d^{g(\mathcal{C})}(b) = \{e\}, N_d^{g(\mathcal{C})}(c) = \{d\}, N_d^{g(\mathcal{C})}(d) = \{c\}, N_d^{g(\mathcal{C})}(e) = \emptyset$. Let $A = \{e\}$. Then, $\underline{A}_{g(\mathcal{C}),2} = \{a, b, e\} \nsubseteq A = \{e\}$.

Rough Set model for IqBa2: Let $\langle U, g(\mathcal{C})\rangle$ be a g-covering approximation space with for each $u \in U$, $N_d^{g(\mathcal{C})}(u)$ contains at most one element of U. Now, $\langle 2^U, \cap, \cup, \neg, \emptyset, U\rangle$ is a qBa, where $\neg A = (g(A))^c$, for all $A \in 2^U$. We define $\rightarrow$ in 2^U as follows
$A \rightarrow B = A^c \cup B$.
Then, it is obvious that $A \rightarrow B = U$ if and only if $A \subseteq B$ and consequently $\langle 2^U, \cap, \cup, \rightarrow, \neg, \emptyset, U\rangle$ becomes an IqBa. We now define IA, for all $A \subseteq U$ as $IA = \underline{A}_{g(\mathcal{C}),2}$. Then by Proposition 6 and Theorem 1, $\langle 2^U, \cap, \cup, \rightarrow, \neg, I, \emptyset, U\rangle$ is an IqBa2.

Remark 2.

1. If we define implication as $A \rightarrow_1 B = g(A \rightarrow B) = \neg A \cup g(B)$, for all $A, B \in 2^U$ then $\langle 2^U, \cap, \cup, \rightarrow_1, \neg, I/I_1, \emptyset, U\rangle$ becomes a different model for IqBa2 with respect to the implication $\rightarrow_1$.
2. By Example 4, modal axiom T: $Ia \leq a$ [8] does not hold and hence $\langle 2^U, \cap, \cup, \rightarrow, \neg, I, \emptyset, U\rangle$ is not a model for IqBa2,T.

4 Conclusion and Future Work

We may summarise the contents of this paper and indicate some future directions of work as follows.

– A g-covering approximation space has been developed out of a covering approximation space and an involution g. A necessary and sufficient condition is obtained so that these two spaces coincide.
– Familiar notions that are available in a covering approximation space have been introduced in a g-covering approximation space and relationships between them are studied.
– Deleted neighborhood or anti-reflexive neighborhood has been incorporated in this theory. Basically, they are not granules but their importance has been mentioned [9] in the field of computer security.
– A pair of lower-upper approximations has been introduced which are dual with respect to the quasi-complementation in a g-covering approximation space. Using them, a rough set model of IqBa2 has been presented.
– In covering based rough set theory, there are many lower-upper approximations of a set in various literature. Some of them are dual with respect to the set-theoretic complementation whereas other pairs are not so. A study may be continued on them so that the notion of quasi-complementation can be incorporated and rough set models of remaining algebras may be constructed.

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